Logarithmic Test

Suppose $\sum u_n$ is series of +ve $\mathbb{R}$ and let

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = m. \quad \text{Then}$$

$\sum u_n$ conv if $m > 1$

$\sum u_n$ divg if $m < 1$

[We will go for log test if ratio test fails & there is a power form]

Example [conv / divg]

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots \qquad (x > 0)$$

Ans : $\sum u_n$ where $u_n = \dfrac{n^n x^n}{n!}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \left(1 + \frac{1}{n}\right)^n x = ex$$

$\Rightarrow$ By ratio test, $\sum u_n$ conv if $0 < x < \frac{1}{e}$

and divg if $x > \frac{1}{e}$

~~If~~ $x = \frac{1}{e}$ ratio test fails.

Using log test,

$$\lim n \log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n \log \frac{en^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} n \left(1 + n \log \frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty} n \left(1 - n \log\left(1 + \frac{1}{n}\right)\right)$$

(1+1. W)

$$= \lim n \left(1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + o\left(\frac{1}{n^3}\right)\right)$$

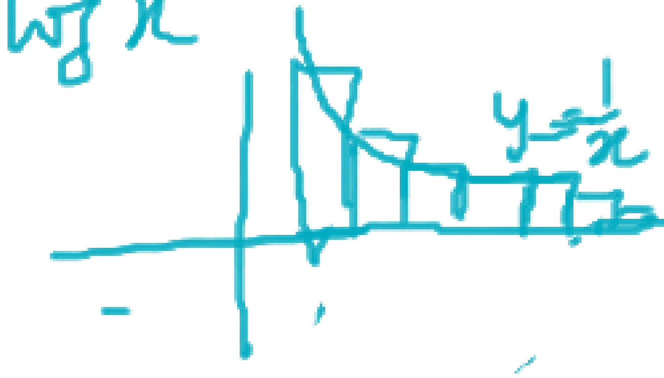
$$= \frac{1}{2}$$

$\Rightarrow \sum u_n$ divg for $x \geq \frac{1}{e}$

H. W. Discuss the conv / divg of the following

(1) $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^2}x^3 + \cdots$

$\log x = \int \frac{1}{x}\,dx$



(2) $x + x^{1+\frac{1}{2}} + x^{1+\frac{1}{2}+\frac{1}{3}} + \cdots$

(3) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{\log n} > \sum \left(\frac{1}{2}\right)^n \rightarrow$ conv

$y = \frac{1}{x}$

(4) $\frac{1}{4} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}} + \cdots$

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Cauchy Condensation Test

Let $\{f(n)\}$ be a decreasing seq of +ve $\mathbb{R}$ and $a > 1$ be an integer. Then $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together.

proof: $\sum f(n) = [f(1)] + [f(2) + \cdots + f(a)]$
$+ [f(a+1) + \cdots + f(a^2)]$
$+ [f(a^2+1) + \cdots + f(a^3)] +$

$$+ \cdots$$

$$+ \left[f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots + f(a^n) \right]$$

$$+ \cdots$$

$$= [f(1)] + [v_1] + [v_2] + \cdots + [v_n] + \cdots$$

$$= f(1) + \sum v_n$$

where $v_n = f(a^{n-1}+1) + \cdots + f(a^n)$

$$\geqslant (a^n - a^{n-1}) \, f(a^n)$$

observe that,

$$(a^n - a^{n-1}) f(a^n)$$

$$= a^n \left(\frac{a-1}{a}\right) f(a^n)$$

$$= \left(\frac{a-1}{a}\right) w_n$$

$$\Rightarrow w_n \leq \frac{a}{a-1} v_n$$

$\Rightarrow \sum w_n$ converges if $\sum v_n$ converges

& $\sum v_n$ diverges if $\sum w_n$ diverges.

Again,

$$v_n = f(a^{n-1}+1) + \dots + f(a^n)$$

$$\leq (a^n - a^{n-1}) \left(f(a^{n-1}+1) \right)$$

$$\leq a^{n-1}(a-1) \, f(a^{n-1})$$

$$= (a-1) w_n$$

$\Rightarrow v_n \leq (a-1) w_n \Rightarrow$ $\sum v_n$ conv if $\sum w_n$ conv & $\sum w_n$ div if $\sum v_n$ div

$\Rightarrow \sum v_n$ & $\sum W_n$ conv / divg together.

Again, $\sum v$, $\sum f(n)$ conv / div together
(H.W.)

$\Rightarrow \sum f(n)$, $\sum W_n$ conv. / divg together

$\Rightarrow \sum f(n)$, $\sum a^n f(a^n)$ conv / divg Together

[Proved]

Examples

① Discuss the convergence of the series $\sum \frac{1}{(n \log n)^p}$

Ans. The given series can be written in the form $\sum f(n)$ where $f(n) = \frac{1}{(n \log n)^p}$

Now, $a^n f(a^n) = \frac{a^n}{(a^n \log a^n)^p}$

$= \frac{1}{a^{(p-1)n} \, n^p (\log a)^p}$

Case $p > 1$

$$a^n f(a^n) < \frac{1}{n^p}$$

Now, $\sum \frac{1}{n^p}$ conv for $p > 1$

$\Rightarrow \sum a^n f(a^n)$ conv $\Rightarrow \sum f(n)$ conv for $p > 1$

Case $p < 1$

$$a^n f(a^n) > \frac{1}{n^p (\log a)^p}$$

Now, $\sum \frac{1}{n^p}$ div for $p<1$. $\Rightarrow \sum a^n f(a^n)$ divg

$\Rightarrow \sum f(n) = \frac{1}{(n \log n)^p}$ div for $p<1$

Case $p=1$

$$a^n f(a^n) = \frac{1}{n(\log a)}$$

$\sum \frac{1}{n}$ div $\Rightarrow \sum a^n f(a^n)$ divg

$\Rightarrow \sum f(n) = \frac{1}{(n \log n)^p}$ divg for $p=1$.

H.W. Discuss conv/divg of following series:

1. $$\sum \frac{1}{n^p}$$

2. $$\sum \frac{1}{n(\log n)^p}$$

3. $$\sum \frac{1}{n \log n \, (\log \log n)}$$

General Series

[This type of series can have both +ve or -ve terms.]

Alternating Series

A series of the form $\sum (-1)^{n-1} a_n$ where $a_n > 0$ is called alternating series

Leibnitz's Test

If $\{u_n\}$ be monotone decreasing seq of +ve $\mathbb{R}$ and $\lim_{n\to\infty} u_n = 0$.

Then $u_1 - u_2 + u_3 - u_4 + \ldots$ is convergent.

Proof: Let $S_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n$

$$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$$

$\Rightarrow S_n \uparrow$. Also, $S_{2n} = u_1 - (u_2 - u_3) - \cdots - u_{2n} < u_1$

$\Rightarrow \{S_{2n}\}$ is $\uparrow$ & bounded above

$\Rightarrow \{S_{2n}\}$ is conv.

Also, $S_{2n+1} - S_{2n-1} = -u_{2n} + u_{2n+1} \leq 0$

$\Rightarrow S_{2n-1} \downarrow$

Again $S_{2n-1} = (u_1 - u_2) + (u_3 - u_4) + \cdots + u_{2n-1}$

$< u_1 - u_2$

$\Rightarrow \{S_{2n-1}\} \downarrow$ & bounded below $\underset{\text{conv}}{\Rightarrow} \{S_{2n-1}\}$

$\Rightarrow$ Both $\{S_{2n}\}$ & $\{S_{2n-1}\}$ conv.

Moreover, $\lim_{n\to\infty} (S_{2n} - S_{2n-1}) = \lim_{n\to\infty} u_{2n} = 0$

$\Rightarrow \{S_{2n-1}\}, \{S_{2n}\}$ conv to same limit.

$\Rightarrow$ Even & Odd subseq both conv to same limit.

$\Rightarrow \{S_n\}$ convergent.

$\Rightarrow u_1 - u_2 + u_3 - u_4 + \cdots$ conv. (Proved)

Imp

Examples

Discuss the conv of the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots\cdots$$

Ans The given series can be written as

$$\sum (-1)^{n-1} u_n \quad \text{where} \quad u_n = \frac{n+1}{n} \qquad \left[\lim u_n = 1\right]$$

Now, $u_{n+1} - u_n = \frac{n+2}{n+1} - \frac{n+1}{n} = -\frac{1}{n(n+1)} < 0$

$\Rightarrow u_{n+1} < u_n \Rightarrow \{u_n\} \downarrow$ But $\lim_{n\to\infty} u_n = l \neq 0$

$\Rightarrow \sum (-1)^{n-1} u_n$ NOT conv.

H.W- Discuss the convergence of the following series

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

(3) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n+a^2}$

H.W. If $\{u_n\}$ be a $\downarrow$ seq of positive $\mathbb{R}$ and $\lim_{n\to\infty} u_n = 0$ then discuss the convergence of following series

$$(1) \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{u_1 + u_2 + \cdots + u_n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{u_1 + u_3 + \cdots + u_{2n-1}}{2n-1}$$